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Singular Limit of a *p*-Laplacian Reaction-Diffusion Equation with a Spatially Inhomogeneous Reaction Term

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We study singular limit of a p-Laplacian reaction-diffusion equation with a spatially inhomogeneous reaction term. The coefficient of the reaction term is much larger than the diffusion coefficient and sharp interfaces appear between two phases. We show by matched asymptotic expansions that the limit equation (interface equation) is a mean curvature flow with drift terms, similar to the case $p=2$.

KEY WORDS: p-Laplacian reaction-diffusion equations; nonlinear diffusion; singular perturbation; interface equations.

1. INTRODUCTION

It is well-known that some classes of nonlinear diffusion equations give rise to sharp internal layers (or interfaces) when the diffusion coefficient is very small or the reaction term is very large. And the motion of such interfaces is often driven by their mean curvature.

To name only a few, Chen[1] gave a rigorous proof on the generation and propagation of interfaces for

$$u_t = \Delta u - \frac{1}{\varepsilon^2} W'(u), \qquad x \in \mathbb{R}^N, \quad t > 0,$$

where $\varepsilon > 0$ is a small parameter and $W(u)$ is a double-well potential of equal well-depth (a typical example is $W(u) = u^2(u^2-1)$). He showed that the equation of motion of the surface (interface equation) is

$$V = -(N-1)\,\kappa,$$

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where V is the normal velocity of the interface and κ is mean curvature of the interface.

In ref. 2, the authors studied a reaction-diffusion equation with a spatially inhomogeneous reaction term:

$$u_t = \Delta u - \frac{1}{\varepsilon^2} h^2(x)\, W'(u), \qquad x \in \mathbb{R}^N, \quad t > 0, \tag{1.1}$$

with h strictly positive. They showed that the interface equation involves a drift term despite the absence of drift in the original equation:

$$V = -(N-1)\,\kappa - \frac{\partial}{\partial n}(\log h), \tag{1.2}$$

where n is the normal unit vector to the surface (outward, for closed interfaces). The existence of stable stationary closed interfaces for (1.2) in the plane was obtained in ref. 3.

Recently, ref. 4 gave a rigorous proof on the generation and propagation of interfaces for

$$u_t = \operatorname{div}(k(x)\, \nabla u) + \frac{h(x)}{\varepsilon^2}\, u(1-u^2), \qquad \text{in} \quad \Omega \times (0, T).$$

For more details of the interface equations, see, for example, the above listed papers and references therein.

In this paper, we consider a p-Laplacian reaction-diffusion equation (Allen–Cahn equation) with a spatially inhomogeneous reaction term:

$$\frac{1}{\varepsilon^{p-2}}\, u_t = \operatorname{div}(a^p(x)\, |\nabla u|^{p-2}\, \nabla u) - \frac{1}{\varepsilon^p}\, b^p(x)\, W'(u), \qquad x \in \mathbb{R}^N, \quad t > 0, \tag{1.3}$$

where $p > 1$ is a constant, $a(x), b(x)$ are smooth and for some $a_0 > 0$, $a(x) \geqslant a_0 (x \in \mathbb{R}^N)$, $b(x) > 0 (x \in \mathbb{R}^N)$. We show by matched asymptotic expansions that the singular limit of (1.3) (that is, the interface equation) is a mean curvature flow with drift terms concerning $\frac{\partial a}{\partial n}$ and $\frac{\partial b}{\partial n}$. Though the drift terms may be not exact gradient times the normal, the interface equation is contained in the class of mean curvature flow with drift terms, just as the case $p = 2$.

2. A FORMAL DERIVATION OF INTERTFACE EQUATIONS

In this section, we present a formal derivation of the equation of motion of interface for Eq. (1.3). The technique is based on matched asymptotic expansions using the so-called signed distance function, which can be found in refs. 2 and 5, etc. We assume that $f(u) = -W'(u)$ is a smooth function derived from a double-well potential $W(u)$ whose local minima lie at $u = -1$ and $u = 1$. Moreover we assume that $f(-1) = f(0) = f(1)$, $f'(-1) < 0$, $f'(0) > 0$, $f'(1) < 0$ and $\int_{-1}^{1} f(s)\, ds = 0$. The last condition is equivalent to $W(-1) = W(1)$.

2.1. Matched Asymptotic Expansions

Let u^ε be a solution of (1.3) and Γ^ε be the interface

$$\Gamma^\varepsilon = \bigcup_{t \geqslant 0} (\Gamma_t^\varepsilon \times \{t\}),$$

where $\Gamma_t^\varepsilon = \{x \in \mathbb{R}^N \mid u^\varepsilon(x, t) = 0\}$. Hereafter, we assume that the interface Γ^ε is smooth and that Γ_t^ε is a smooth closed hypersurface in $\mathbb{R}^N$ without boundaries for each $t \geqslant 0$. We denote by Ω_t^ε the bounded domain in $\mathbb{R}^N$ enclosed by Γ_t^ε.

Let $d^\varepsilon(x, t)$ be the signed distance function to Γ^ε defined by

$$d^\varepsilon(x, t) = \begin{cases} \operatorname{dist}(x, \Gamma_t^\varepsilon), & x \in \mathbb{R}^N \setminus \overline{\Omega_t^\varepsilon}, \\ -\operatorname{dist}(x, \Gamma_t^\varepsilon), & x \in \Omega_t^\varepsilon. \end{cases} \tag{2.1}$$

We remark that $d^\varepsilon = 0$ on Γ^ε and $|\nabla d^\varepsilon| = 1$. We assume that d^ε has the expansion

$$d^\varepsilon(x, t) = d_0(x, t) + \varepsilon d_1(x, t) + \varepsilon^2 d_2(x, t) + \cdots$$

and denote

$$\Gamma_t = \{x \in \mathbb{R}^N \mid d_0(x, t) = 0\}, \qquad \Omega_t = \{x \in \mathbb{R}^N \mid d_0(x, t) < 0\}$$

$$\Gamma = \bigcup_{t \geqslant 0} (\Gamma_t \times \{t\}), \qquad Q_0 = \bigcup_{t \geqslant 0} ((\mathbb{R}^N \setminus \overline{\Omega_t}) \times \{t\}), \qquad Q_1 = \bigcup_{t \geqslant 0} (\Omega_t \times \{t\}).$$

Roughly speaking, Γ_t represents the position of the interface at time t in the limit as $\varepsilon \to 0$, while Ω_t represents the region inside Γ_t.

We also assume that the solution u^ε has the expansions

$$u^\varepsilon(x, t) = u_0(x, t) + \varepsilon u_1(x, t) + \varepsilon^2 u_2(x, t) + \cdots \tag{2.2}$$

away from the interface Γ^ε (the outer expansion) and

$$u^\varepsilon(x, t) = U_0(\xi, x, t) + \varepsilon U_1(\xi, x, t) + \varepsilon^2 U_2(\xi, x, t) + \cdots \tag{2.3}$$

near Γ^ε (the inner expansion), where $\xi = d^\varepsilon(x,t)/\varepsilon$. The stretched space variable ε gives exactly the right spatial scaling to describe the sharp transition between the regions $\{u \approx -1\}$ and $\{u \approx 1\}$. Since $u^\varepsilon = 0$ on Γ^ε, we normalize U_k in such a way that $U_k(0,x,t) = 0 (k = 0,1,2,...)$ for all (x,t) near Γ^ε (normalization conditions). To make the inner and outer expansions consistent, we require that

$$U_k(+\infty, x, t) = u_k^+(x,t) \qquad \text{if} \quad x \in (\mathbb{R}^N \backslash \overline{\Omega_t}) \cup \Gamma_t \tag{2.4}$$

$$U_k(-\infty, x, t) = u_k^-(x,t) \qquad \text{if} \quad x \in \Omega_t \cup \Gamma_t \tag{2.5}$$

for all (x,t) near Γ and all $k \geqslant 0$ (matching conditions), where u_k^+ and u_k^- respectively denote the terms of outer expansion (2.2) in the region Q_0 and the region Q_1. In particular, if $x \in \Gamma_t$, then one has to take into account both of the conditions (2.4) and (2.5).

2.2. Motion of Interface for Equation (1.3)

Substituting the outer expansion (2.2) into (1.3) and collecting the ε^{-p} and ε^{-p+1} terms respectively, we have

$$b^p(x)\, f(u_0(x,t)) = 0, \qquad b^p(x)\, f'(u_0(x,t))\, u_1(x,t) = 0.$$

in $Q_0 \cup Q_1$. Hence we have $u_0 = -1$, $u_0 = 0$ or $u_0 = 1$ in $Q_0 \cup Q_1$. Since we are studying interfaces between the region $\{u \approx -1\}$ and $\{u \approx 1\}$, we have either $u_0(x,t) = -1$ in Q_0 and $u_0(x,t) = 1$ in Q_1 or the other way around. As both cases are treated similarly, we will assume the former throughout this section. From the second equality we get $u_1(x,t) = 0$ in $Q_0 \cup Q_1$.

Now, substituting (2.3) into (1.3) and collecting the ε^{-p} and ε^{-p+1} terms, we get

$$a^p(x)\, U_{0\xi}^{p-2} U_{0\xi\xi} + b^p(x)\, f(U_0) = 0, \tag{2.6}$$

$$\begin{aligned} U_{0\xi} d_{0t} = {} & U_{0\xi}^{p-1}\, \nabla d_0 \cdot \nabla a^p + (p-2)\, a^p U_{0\xi}^{p-3} U_{0\xi\xi}\, \nabla d_0 \cdot \nabla_x U_0 \\ & + (p-2)\, a^p U_{0\xi}^{p-3} U_{0\xi\xi} U_{1\xi} + 2a^p U_{0\xi}^{p-2}\, \nabla d_0 \cdot \nabla_x U_{0\xi} \\ & + a^p U_{0\xi}^{p-1} \Delta d_0 + a^p U_{0\xi}^{p-2} U_{1\xi\xi} + b^p f'(U_0)\, U_1 \end{aligned} \tag{2.7}$$

(2.7) is in fact

$$\begin{aligned} & a^p U_{0\xi}^{p-2} U_{1\xi\xi} + (p-2)\, a^p U_{0\xi}^{p-3} U_{0\xi\xi} U_{1\xi} + b^p f'(U_0)\, U_1 \\ & \quad = U_{0\xi} d_{0t} - U_{0\xi}^{p-1} \nabla d_0 \cdot \nabla a^p - (p-2)\, a^p U_{0\xi}^{p-3} U_{0\xi\xi} \nabla d_0 \cdot \nabla_x U_0 \\ & \quad\quad - 2a^p U_{0\xi}^{p-2} \nabla d_0 \cdot \nabla_x U_{0\xi} - a^p U_{0\xi}^{p-1} \Delta d_0 \end{aligned} \tag{2.8}$$

Suppose that $\varphi(\xi)$ is the unique solution of

$$\begin{cases} (\varphi')^{p-2}\varphi'' + f(\varphi) = 0, \\ \varphi(-\infty) = -1,\ \varphi(0) = 0,\ \varphi(+\infty) = 1. \end{cases} \tag{2.9}$$

Hence

$$\varphi'(\xi) = [p(W(\varphi(\xi)) - W(-1))]^{1/p} > 0, \qquad \xi \in \mathbb{R}.$$

Therefore

$$U_0(\xi, x, t) = \varphi\left(\frac{b(x)}{a(x)}\,\xi\right) \tag{2.10}$$

for $\xi \in \mathbb{R}$ and all (x, t) near Γ, is the solution of (2.6). Substituting (2.10) into (2.8) we have

$$\begin{aligned} &a^2 b^{p-2}(\varphi')^{p-2}\, U_{1\xi\xi} + (p-2)\, a b^{p-1}(\varphi')^{p-3}\, \varphi'' U_{1\xi} + b^p f'(\varphi)\, U_1 \\ &\quad = \frac{b}{a}\,\varphi' d_{0t} - \frac{b^{p-1}}{a^{p-1}}\,(\varphi')^{p-1}\, \nabla d_0 \cdot \nabla a^p - (p-2)\, a b^{p-1} \xi (\varphi')^{p-2}\, \varphi''\, \nabla d_0 \cdot \nabla\left(\frac{b}{a}\right) \\ &\qquad - 2a^2 b^{p-2}(\varphi')^{p-2}\, \nabla d_0 \cdot \left(\varphi' \nabla\left(\frac{b}{a}\right) + \frac{b}{a}\,\xi \varphi'' \nabla\left(\frac{b}{a}\right)\right) - a b^{p-1}(\varphi')^{p-1}\, \Delta d_0 \end{aligned}$$

Let $z = \frac{b}{a}\,\xi$ and taking into consideration the normalization condition we have

$$\begin{aligned} &b^p(\varphi')^{p-2}\, U_{1zz} + (p-2)\, b^p(\varphi')^{p-3}\, \varphi'' U_{1z} + b^p f'(\varphi)\, U_1 \\ &\quad = \frac{b}{a}\,\varphi' d_{0t} - p b^{p-1}(\varphi')^{p-1}\, \nabla d_0 \cdot \nabla a - p a^2 b^{p-2}\, \nabla d_0 \cdot \nabla\left(\frac{b}{a}\right) \cdot z(\varphi')^{p-2}\, \varphi'' \\ &\qquad - 2a^2 b^{p-2}(\varphi')^{p-1}\, \nabla d_0 \cdot \nabla\left(\frac{b}{a}\right) - a b^{p-1}(\varphi')^{p-1}\, \Delta d_0 \\ &\quad \equiv A(z, x, t) \end{aligned} \tag{2.11}$$

$$U_1(0, x, t) = 0.$$

Now we use a Fredholm type lemma like Lemma 4.1 in ref. 6 or Lemma 2.1 in ref. 2. Observing that $\psi = \varphi'$ solves

$$\psi^{p-2}\psi'' + (p-2)\, \psi^{p-3}(\psi')^2 + f'(\varphi)\, \psi = 0$$

and using the method of variation of constants one can find the explicit solution of (2.11): $U_1(z, x, t) = \alpha(z, x, t)\, \psi(z)$, where α satisfies

$$\psi^{p-1}\alpha_{zz} + p\psi^{p-2}\psi'\alpha_z = A(z, x, t)/b^p,$$

i.e.,

$$(\alpha_z \psi^p)_z = \psi A/b^p. \tag{2.12}$$

A solution α of (2.12) exists if and only if (see, e.g., Lemma 4.1 in ref. 6)

$$\int_{\mathbb{R}} A(z, x, t)\, \psi(z)\, dz = 0.$$

Substituting the right hand side of $(2.11)_a$ into this equality and noticing $\int_{\mathbb{R}} z(\varphi')^{p-1}\, \varphi''\, dz = -(1/p) \int_{\mathbb{R}} (\varphi')^p\, (z)\, dz$, we obtain

$$\frac{g_2 b}{a}\, d_{0t} = ab^{p-1} g_p \cdot \Delta d_0 + ab^{p-2} g_p\, \nabla b \cdot \nabla d_0 + (p-1)\, b^{p-1} g_p\, \nabla a \cdot \nabla d_0, \tag{2.13}$$

where

$$g_j = \int_{\mathbb{R}} (\varphi'(z))^j\, dz \qquad \forall j = 1, 2, \ldots, p.$$

are constants depend on j.

Since ∇d_0 coincides with the outward normal unit vector n to the hypersurface Γ_t, and $-d_{0t}(x, t) = V$, where V is the normal velocity of the interface Γ_t. It is also known that the mean curvature κ of the interface is $\frac{\Delta d_0}{N-1}$. Thus (2.13) is equivalent to

$$V = -\frac{g_p}{g_2}\left[(N-1)\, a^2 b^{p-2}\kappa + a^2 b^{p-3} \frac{\partial b}{\partial n} + (p-1)\, ab^{p-2} \frac{\partial a}{\partial n} \right]. \tag{2.14}$$

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REFERENCES

1. X. Chen, Generation and propagation of interfaces for reaction-diffusion equations, *J. Differential Equations*, **96**:116–141 (1992).

2. K.-I. Nakamura, H. Matano, D. Hilhorst, and R. Schatzle, Singular limit of a reaction-diffusion equations with a spatially inhomogeneous reaction term, *J. Statist. Phys.* **95**:1165–1185 (1999).
3. Y. Oshita and B. Lou, Interfaces in a generalized mean curvature flow with nonconstant driving force, *Submitted.*
4. D. Hilhorst, H. Matano, and R. Schätzle, Singular limits of an inhomogeneous reaction-diffusion equation, *Preprint*.
5. P. C. Fife, *Dynamics of Internal Layers and Diffusive Interfaces*, CBMS-NSF Regional Conf. Ser. in Appl. Math. (SIAM, Philadephia, 1988).
6. N. D. Alikakos, P. W. Bates, and X. Chen, Convergence of the Cahn–Hilliard equation to the Hele–Shaw model, *Arch. Rational Mech. Anal.* **128**:165–205 (1994).